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Transfer matrices for the totally asymmetric simple exclusion process

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Abstract
We consider the totally asymmetric simple exclusion process on a finite lattice with open boundaries. We show, using the recursive structure of the Markov matrix that encodes the dynamics, that there exist two transfer matrices $T_{L-1,L}$ and $\tilde{T}_{L-1,L}$ that intertwine the Markov matrices of consecutive system sizes: $\tilde{T}_{L-1,L}M_{L-1} = M_L T_{L-1,L}$. This semi-conjugation property of the dynamics provides an algebraic counterpart for the matrix-product representation of the steady state of the process.

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1. Introduction

Nonequilibrium statistical mechanics has progressed a lot of thanks to the study of interacting particle processes in one dimension [1–3]. In this field, the asymmetric simple exclusion process (ASEP) has been playing a paradigmatic role with an impressive body of knowledge accumulated during the last 20 years [4–7]. This model can be investigated from very different points of view: simple exclusion indeed has a complex story [8]. Two noteworthy techniques are the Bethe Ansatz [9, 10], useful for spectral properties of the dynamics, and the matrix Ansatz, initiated in [11], in which the stationary measure of the model is represented as a matrix-product state. The matrix Ansatz first appeared in the study of the totally asymmetric simple exclusion process (TASEP) on a finite discrete lattice with open boundaries, as a trick to represent the stationary weights of the configurations. It was observed, empirically, [12, 13] that the weights of configurations of sizes L and $L-1$ are related through recursion relations that can be suitably encoded in a matrix-product state. Subsequently, this idea bloomed into a fruitful and powerful technique that can be summarized as follows: given a stochastic model, look for a suitable algebra to represent its steady state. A recent and exhaustive review of the matrix-product representation for stochastic nonequilibrium systems can be found in [14].

However, the very fact that the weights of configurations of *different* sizes can be related through some combinatorial identities is deeply puzzling; indeed, two models of different sizes have *a priori* no relation at all with one another: their phase spaces are totally disconnected (the dynamics conserves the size of the system). Moreover, the matrix Ansatz does not seem to be logically related to the structure of the Markov operator: the algebra used to represent the stationary weights has often to be determined by inspection of simple cases or by analogy with known examples [15–20] (the review [14] that describes in detail most of the exact solutions in the field is particularly helpful in this respect). Once the algebra is found, the steady state is written as a trace over this algebra and is shown to vanish under the action of the Markov matrix. This last step of the proof involves a cancellation mechanism usually requiring an auxiliary ('hat') algebra [21, 22]. It has been shown rigorously that most of the models admit formally a matrix Ansatz but, unfortunately, the proof is not constructive [23].

The aim of the present work is to show for the TASEP with open boundaries that the Markov matrices of two consecutive system sizes are semi-conjugate of each other through two transfer matrices. This conjugation property is a characteristic of the dynamics and it only relies on the recursive structure of the Markov matrix. A similar property has been proved for the multi-species exclusion process on a ring for which the matrix-product representation involves complicated tensor products of quadratic algebras [24–26]. Recently, a conjugation property has been used to derive exact results for an annihilation model for which a matrix Ansatz could not be found [27]. We believe that the existence of a dynamical conjugation that relates a given model to a simpler one (simpler because it involves a smaller number of sites, or of particles, or of types of particles, etc) is a fundamental property that underlies the solvability of many nonequilibrium processes.

2. Recursive structure of the TASEP Markov matrix

We first recall the dynamical rules of the TASEP in a one-dimensional discrete lattice of size L with open boundaries. In this model, particles are injected at site 1 at rate α and rejected from site L at rate β. Every site can be occupied by at most one particle. Particles in the bulk can hop stochastically with rate 1 from a site to the adjacent site on its right if it is vacant. The phase space Ω_L of the system consists of 2^L different configurations.

The Markov matrix for a general stochastic system on a 1D lattice of size L with open boundaries and nearest-neighbour interactions is given by [23]

$$M_L = h_L^{\text{left}} + \sum_{i=1}^{L-1} h_L(i, i+1) + h_L^{\text{right}}, \tag{1}$$

with

$$h_L^{\text{left}} = h^{(l)} \otimes \mathbb{1}_{L-1}, \qquad h_L^{\text{right}} = \mathbb{1}_{L-1} \otimes h^{(r)}, \tag{2}$$

$$h_L(i, i+1) = \mathbb{1}_{i-1} \otimes h \otimes \mathbb{1}_{L-i-1}. \tag{3}$$

Here, we restrict ourselves to a two-dimensional state space. Thus, $h^{(l)}$ and $h^{(r)}$ are 2×2 matrices reflecting the boundary interactions and h is the local 4×4 matrix for the bulk. More precisely, we have for the TASEP

$$h^{(l)} = \begin{pmatrix} -\alpha & 0 \\ \alpha & 0 \end{pmatrix}, \qquad h^{(r)} = \begin{pmatrix} 0 & \beta \\ 0 & -\beta \end{pmatrix} \qquad \text{and}$$
$$h = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4}$$

Throughout this paper we use $\mathbb{1}_d$ for the 2^d-dimensional identity, but for $d = 1$ the index is suppressed. We observe that generally the following recursion for the Markov matrix is satisfied:

$$M_L = \mathbb{1} \otimes M_{L-1} - \mathbb{1} \otimes h_{L-1}^{\text{left}} + h_{L-1}^{\text{left}} \otimes \mathbb{1} + h_{L-1}(1,2) \otimes \mathbb{1}. \tag{5}$$

We now explain this formula: a system of size L can be obtained by adding a site with index 0 to the system of size $L - 1$ between the left reservoir and site 1. Then, the Markov matrix M_L for the larger system can be expressed in terms of M_{L-1}. Naively, one would write $M_L = \mathbb{1} \otimes M_{L-1}$; this is correct in the bulk but not in the vicinity of the left boundary. Therefore, this incorrect formula has to be repaired as follows: (i) particles now enter at site 0, so we must subtract the matrix elements that correspond to injecting particles at site 1 which no longer occurs (second term on the rhs). (ii) We add a matrix which makes the particles enter at site 0 (third term). (iii) Finally, particle hopping from site 0 to site 1 is implemented by the fourth term (which is equal to $h \otimes \mathbb{1}_{L-2}$).

For the TASEP, substituting the explicit expressions (4) in equation (5), we are led to the following recursive structure of the Markov matrix:

$$M_L = \begin{pmatrix} M_{L-1} - \nu_{L-1} & \rho_{L-1} \\ \alpha \mathbb{1}_{L-1} & M_{L-1} + \omega_{L-1} \end{pmatrix}, \tag{6}$$

where we have defined

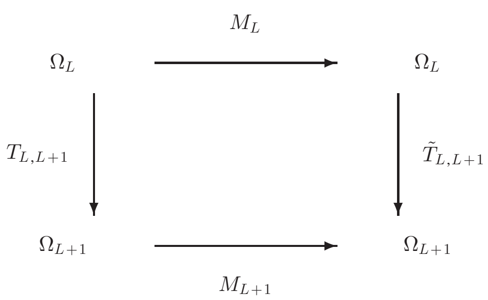
$$\nu_L = \begin{pmatrix} 0 & 0 \\ \alpha \mathbb{1}_{L-1} & \alpha \mathbb{1}_{L-1} \end{pmatrix}, \qquad \rho_L = \begin{pmatrix} 0 & 0 \\ \mathbb{1}_{L-1} & 0 \end{pmatrix} \qquad \text{and}$$
$$\omega_L = \begin{pmatrix} (\alpha - 1)\mathbb{1}_{L-1} & 0 \\ -\alpha \mathbb{1}_{L-1} & 0 \end{pmatrix}. \tag{7}$$

3. Transfer matrices for the TASEP

We shall prove that the Markov matrices M_L and M_{L+1} for two consecutive system sizes are related by the relation

$$\tilde{T}_{L,L+1} M_L = M_{L+1} T_{L,L+1}. \tag{8}$$

This semi-conjugation relation can be illustrated by the following commutative diagram:



We recall that Ω_L and Ω_{L+1} are the phase spaces for the systems of sizes L and $L+1$, respectively. The fact that the diagram is commutative means that there are two equivalent paths to evolve from a state $\mathcal{C}_L$ in Ω_L: either one first applies the dynamics M_L and then identifies the result to a state in Ω_{L+1} via $\tilde{T}_{L,L+1}$, or one imbeds $\mathcal{C}_L$ in Ω_{L+1} via $T_{L,L+1}$ and then applies the dynamics M_{L+1} in the larger phase space. It is important to note that the two transfer matrices T and $\tilde{T}$ are different. If they were equal, then a whole fraction of the spectrum of M_L would be contained in M_{L+1} [26] and this is not true for the TASEP with open boundaries as can be verified with simple explicit cases: for the interesting physical region $\alpha, \beta > 0$, the only common eigenvalue shared by M_L and M_{L+1} is 0, corresponding to the steady state (for generic values of α, β and L). The fact that the two transfer matrices are different is in contrast with the annihilation model studied in [27] where $T = \tilde{T}$. Furthermore, note that $T^{(2)}$ has generically full rank and is therefore invertible while $\tilde{T}^{(2)}$ is not.

Nevertheless, the very existence of this commutative diagram expresses the fact that the dynamics for sizes L and $L+1$ are semi-conjugate to each other. This property also allows us to construct recursively the steady state of a system of size $L+1$ knowing that of the system of size L. If $M_L|v_L\rangle = 0$, the vector $|v_{L+1}\rangle$ defined as

$$|v_{L+1}\rangle = T_{L,L+1}|v_L\rangle \tag{9}$$

satisfies, using (8),

$$M_{L+1}|v_{L+1}\rangle = 0\,. \tag{10}$$

Hence, if $|v_{L+1}\rangle$ is not the null-vector, it is (but for a normalization factor) the stationary state of M_{L+1}. This justifies the name 'transfer matrix' for $T_{L,L+1}$. We also remark that $\tilde{T}_{L,L+1}$ has played no role in this construction. One could use $\tilde{T}$ as a transfer matrix for the left ground state, but for the Markov matrix it is known that the left ground state is the line-vector with all components equal to 1.

We finally emphasize that the commutative diagram that encodes the semi-conjugation property is an intrinsic characteristic of the dynamics and the knowledge of the steady state of the system is not required. The transfer matrices can be found for small systems by solving a linear system and their existence relies on the recursive structure of the Markov matrix itself.

4. Proof of the semi-conjugation property

In this section, we prove relation (8) by constructing explicitly the transfer matrices T and $\tilde{T}$. The transfer matrix $T_{L,L+1}$ for $L \geqslant 1$ is given by

$$T_{L,L+1} = \begin{pmatrix} \frac{1}{\alpha}\mathbb{1}_L \\ T^{(2)}_{L,L+1} \end{pmatrix} \qquad \text{with} \qquad T^{(2)}_{L,L+1} = \begin{pmatrix} \mathbb{1}_{L-1} & \mathbb{1}_{L-1} \\ 0 & T^{(2)}_{L-1,L} \end{pmatrix} \qquad \text{and} \qquad T^{(2)}_{01} := \frac{1}{\beta}. \tag{11}$$

The matrix $T_{L,L+1}$ has been constructed to mimic the recursive algorithm provided by the quadratic algebra found in [11]. We also give an expression for $\tilde{T}_{L,L+1}$ in terms of an unknown square matrix R_{L-1}:

$$\tilde{T}_{L,L+1} = \begin{pmatrix} \frac{1}{\alpha}\mathbb{1}_L \\ \tilde{T}^{(2)}_{L,L+1} \end{pmatrix} \qquad \text{with} \qquad \tilde{T}^{(2)}_{L,L+1} = \begin{pmatrix} \mathbb{1}_{L-1} & \mathbb{1}_{L-1} \\ \alpha R_{L-1} & -R_{L-1}(M_{L-1} - \nu_{L-1}) \end{pmatrix}. \tag{12}$$

Using these expressions of T and $\tilde{T}$, we calculate the left-hand side and the right-hand side of equation (8). Both sides are block rectangular matrices of size 4 by 2 with elements given in terms of the matrices at the level $L - 1$. In order to satisfy equation (8), the eight elements of the lhs matrix must be equal to the eight elements of the rhs matrix. Amongst the eight conditions thus obtained, six are tautologically true. The seventh relation is given by

$$\rho_{L-1} T^{(2)}_{L-1,L} = \frac{\nu_{L-1}}{\alpha}. \tag{13}$$

This equation is easily verified by induction by using (6), the explicit expressions (7) of ρ_{L-1}, ν_{L-1} and that of $T^{(2)}_{L-1,L}$ given in equation (11). The eighth relation to be satisfied is given by

$$\mathbb{1}_{L-1} + (M_{L-1} + \omega_{L-1})\, T^{(2)}_{L-1,L} = R_{L-1}[\alpha\rho_{L-1} - (M_{L-1} - \nu_{L-1})(M_{L-1} + \omega_{L-1})]. \tag{14}$$

This relation can be interpreted as a definition of the unknown matrix R_{L-1}. In other words, the semi-conjugation property (8) will be proved if we are able to construct a family of matrices R that satisfy equation (14) for each system size. We can rewrite the generic equation for the unknown R_L as

$$G_L = R_L H_L, \tag{15}$$

where we have defined

$$\begin{aligned} G_L &= \mathbb{1}_L + (M_L + \omega_L)\, T^{(2)}_{L,L+1} \qquad \text{and} \\ H_L &= \alpha\rho_L - (M_L - \nu_L)\,(M_L + \omega_L) = \nu_L M_L - M_L\,(M_L + \omega_L)\,, \end{aligned} \tag{16}$$

the last equality resulting in the fact that $0 = \alpha\rho_L + \nu_L\omega_L$ as seen from equation (7).

Using the finer structure (6) of the Markov matrix and the expression for $T^{(2)}_{L,L+1}$ given in equation (11), we deduce the following recursion for G:

$$G_{L+1} = \begin{pmatrix} M_L - \nu_L + \alpha\mathbb{1}_L & M_L + \omega_L + \rho_L \\ 0 & G_L \end{pmatrix}, \tag{17}$$

which can be iterated to obtain a well-defined upper triangular matrix for G. Therefore, all the G_L's are known. It remains to extract R_L from equation (15). It is important to note that there is no need to calculate R explicitly since $\tilde{T}$ plays no role in constructing the stationary state (which is entirely determined by the right transfer matrix T): the only thing to prove is that R_L exists, i.e. that the equation $G_L = R_L H_L$ has at least one solution. Thus, equation (15) must satisfy a solvability condition which here amounts to the fact that any vector in the (right) kernel of H_L must also belong to the (right) kernel of G_L, i.e. any ket $|h_L\rangle$ such that $H_L|h_L\rangle = 0$ must satisfy $G_L|h_L\rangle = 0$. By studying explicitly systems of small sizes, we found that the matrix H_L is not invertible and that its kernel consists only of a one-dimensional vector-space. We also checked that this one-dimensional vector-space is included in the kernel of G_L (which is of dimension $2^L - L - 1$). Formal proofs of these facts are given in the appendix. This shows that R_L exists and concludes the proof of the semi-conjugation property (8).

5. Conclusions

To summarize, we have shown that for the TASEP with open boundaries, the Markov matrices M for consecutive system sizes are related to each other by a semi-conjugation relation (8) via two different transfer matrices T and $\tilde{T}$. An explicit form of T is obtained from the matrix-product Ansatz and the existence of the left transfer matrix $\tilde{T}$ is proved. This relation between the dynamics corresponding to two consecutive system sizes is a consequence of the recursive structure of the Markov matrix. We believe that this correspondence expresses a fundamental property which is at the heart of the exact solutions of many nonequilibrium models. More precisely, from a formal point of view, one can always construct the T matrices as in equation (8), simply by mapping the stationary state of a system of size L into that of size $L + 1$. However, the solvability of the system is expressed by the fact that the T matrices obey some 'simple' recursion relations. Given a certain process, the challenge is to find a non-trivial transfer matrix T that gives the steady state of the next higher system (without actually presuming it). In fact, the same feature also occurs for the matrix-product representation: one can always express the stationary weights of any system as a matrix product; this is simply a way of encoding data (this result is known as the 'Krebs–Sandow' theorem [23]). But in general these matrices satisfy no special algebraic relations, they depend on the size of the system and there are no *a priori* rules to construct them. Finding these algebraic relations or a systematic way of constructing the matrices amounts to solving exactly the problem. The same is true for the semi-conjugation relations. We also emphasize that the case where $T = \tilde{T}$ is very special and involves the embedding of the spectrum of M_L in M_{L+1} [26, 27], meaning that the whole dynamics of a given system can be mapped into the dynamics of the larger system. This finer property is not true for the TASEP with open boundaries.

The existence of semi-conjugation expressed by equation (8) can be investigated on small system sizes by solving a set of linear equations, whereas we do not know how to test the existence of a matrix-product representation on small system sizes. Above all, we were curious to understand the relation between the recursive structure of the Markov matrix and the one implied by the matrix-product representation. It seems to us that the present work gives a partial answer to this question. Transfer matrices do appear in other related models such as the multi-species ASEP on a ring [25], the recently studied annihilation model with exclusion [27] and for discrete-time dynamics such as ordered sequential and fully parallel update [28]. It would also be of interest to extend this approach to the partially asymmetric exclusion process with open boundaries.

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Appendix

A.1. The matrix Ansatz for the TASEP

A configuration $\mathcal{C}$ of the TASEP on a discrete lattice of L sites can be specified by assigning the binary values of L local variables τ_i with $\tau_i = 1$ if site i is occupied and $\tau_i = 0$ otherwise. In the long time limit, the TASEP reaches a steady state with a non-trivial stationary measure: the stationary probability $p(\mathcal{C})$ of the configuration $\mathcal{C}$ can be written as a matrix element over an ordered product of L matrices E and D representing empty and occupied sites respectively:

$$p(\mathcal{C}) = \frac{1}{Z_L}\langle W|\prod_{i=1}^{L}(\tau_i D + (1-\tau_i)E)\,|V\rangle, \tag{A.1}$$

where Z_L is a normalization factor. As shown in [11], the weights defined in (A.1) correspond to the steady-state probabilities if the operators E and D satisfy, along with the left and right boundary vectors $\langle W|$, $|V\rangle$, the following algebra:

$$DE = D + E, \tag{A.2}$$

$$\langle W|E = \alpha^{-1}\langle W| \tag{A.3}$$

$$D|V\rangle = \beta^{-1}|V\rangle. \tag{A.4}$$

Thanks to these reduction relations, the weight of a configuration of size L can be expressed as a linear combination of the weights of some configurations of size $L-1$. This matrix-product representation yields exact formulae for the currents, the density profiles and the steady correlations, and allows us to determine the exact phase diagram of the model [11]. This algebra implies the formula (11) for the transfer matrix: the idea is to always apply rule (A.3) if possible; with second priority apply (A.2) and if this too is not possible then apply (A.4).

A.2. Proof that the Kernel of H_L is one dimensional

The matrix H_L, of size 2^L, is defined in (16). It can be considered as the 'unevaluated determinant' of the matrix M_{L+1}, given in equation (6). More precisely, we have

$$\det M_{L+1} = \det(-H_L) = \det H_L\,. \tag{A.5}$$

To write this equation, we treat M_{L+1} as a 2 by 2 block matrix whose elements are themselves matrices of size 2^L and we apply the following theorem, proved in [29]: if A, B, C and D are square matrices and if C and D commute, then

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC)\,. \tag{A.6}$$

Here $C = \alpha\mathbb{1}_L$. More generally, using the same theorem, the characteristic polynomial $\pi(X)$ of H_L can be written as

$$\pi(X) = \det(X\mathbb{1}_L - H_L) = \det\left[M_{L+1} - \frac{X}{\alpha}\begin{pmatrix} 0 & \mathbb{1}_L \\ 0 & 0 \end{pmatrix}\right]. \tag{A.7}$$

The kernel of H_L is of dimension 1 if $\pi(X)$ is divisible by X but not by X^2. We show now that for small X, $\pi(X)$ is of order X. We recall that $\pi(X)$ is given by the product of the eigenvalues of the matrix $\left[M_{L+1} - \frac{X}{\alpha}\left(\begin{smallmatrix} 0 & \mathbb{1}_L \\ 0 & 0 \end{smallmatrix}\right)\right]$. For $X = 0$, we know that the spectrum of M_{L+1} consists of the eigenvalue 0 (with multiplicity 1) and that all the other eigenvalues have strictly negative real parts (Perron–Frobenius theorem, see e.g. [30]). For X sufficiently small, the non-zero eigenvalues will remain away from 0; the zero eigenvalue of M_{L+1} will now be given by $E_0(X)$ that can be calculated at first order in X using perturbation theory:

$$E_0(X) = -\frac{X}{\alpha}\,\langle 0|\begin{pmatrix} 0 & \mathbb{1}_L \\ 0 & 0 \end{pmatrix}|0\rangle + \mathcal{O}(X^2), \tag{A.8}$$

where $\langle 0| = (1, 1, \ldots, 1)$ and $|0\rangle$ are the left and right eigenvectors of M_{L+1} associated with the eigenvalue 0 (i.e. $|0\rangle$ is the steady-state probability vector). Both these vectors have strictly positive entries and therefore the dominant contribution to $E_0(X)$ is of order X. We have shown that the expansion of the characteristic polynomial $\pi(X)$ for small X begins with a term of order X with a non-vanishing coefficient: this proves that 0 is a simple eigenvalue of H_L.

A.3. Explicit construction of the Kernel of H_L

Let us call $\{h_L(\tau_1, \ldots, \tau_L)\}$ the components of a non-zero vector h_L in the Kernel of H_L. These numbers can be obtained in a matrix-product form:

$$h_L(\tau_1, \ldots, \tau_L) = \langle\tilde{W}|\prod_{i=1}^{L}[\tau_i D + (1-\tau_i)E]\,|V\rangle, \tag{A.9}$$

where D, E and $|V\rangle$ are the same as in (A.2)–(A.4) and $\langle\tilde{W}| = \langle W|D$. One has to show that H_L acting on h_L gives the null-vector. The vector h_L can be written as a column-vector,

$$h_L = \begin{pmatrix} \langle\tilde{W}|E\mathcal{S}_{L-1}|V\rangle \\ \langle\tilde{W}|D\mathcal{S}_{L-1}|V\rangle \end{pmatrix} \tag{A.10}$$

where $\mathcal{S}_{L-1}$ stands for all possible strings made of $L-1$ symbols D and E. Then, we have

$$M_L \begin{pmatrix} \langle\tilde{W}|E\mathcal{S}_{L-1}|V\rangle \\ \langle\tilde{W}|D\mathcal{S}_{L-1}|V\rangle \end{pmatrix} = \begin{pmatrix} \langle\tilde{W}|\mathcal{S}_{L-1}|V\rangle - \alpha\langle\tilde{W}|E\mathcal{S}_{L-1}|V\rangle \\ \alpha\langle\tilde{W}|E\mathcal{S}_{L-1}|V\rangle \ - \langle\tilde{W}|\mathcal{S}_{L-1}|V\rangle \end{pmatrix}. \tag{A.11}$$

This equation is derived by considering the generic expression $\mathcal{S}_{L-1} = D^{n_1}E^{n_2}\cdots E^{n_k}$ (with $n_1 + \cdots + n_k = L-1$). We then observe, as in [31], that through the action of M_L all terms in the bulk cancel out (thanks to the algebra $DE = D + E$), all terms on the right boundary cancel out (because of the rule (A.4)) and only the left boundary terms do not simplify because the bra-vector is $\langle\tilde{W}|$ instead of $\langle W|$. It is important to remark that the precise form of $\langle\tilde{W}|$ has not played any role in the derivation of equation (A.11) and that this relation would remain true if $\langle\tilde{W}|$ were replaced by an arbitrary bra-vector.

From this result, we deduce, using equation (7), that

$$\nu_L M_L \begin{pmatrix} \langle\tilde{W}|E\mathcal{S}_{L-1}|V\rangle \\ \langle\tilde{W}|D\mathcal{S}_{L-1}|V\rangle \end{pmatrix} = 0\,. \tag{A.12}$$

Besides, we have

$$\begin{aligned}(M_L + \omega_L) \begin{pmatrix} \langle\tilde{W}|E\mathcal{S}_{L-1}|V\rangle \\ \langle\tilde{W}|D\mathcal{S}_{L-1}|V\rangle \end{pmatrix} &= \begin{pmatrix} \langle\tilde{W}|\mathcal{S}_{L-1}|V\rangle - \langle\tilde{W}|E\mathcal{S}_{L-1}|V\rangle \\ -\langle\tilde{W}|\mathcal{S}_{L-1}|V\rangle \end{pmatrix} \\ &= -\begin{pmatrix} \langle W|E\mathcal{S}_{L-1}|V\rangle \\ \langle W|D\mathcal{S}_{L-1}|V\rangle \end{pmatrix},\end{aligned} \tag{A.13}$$

where we have used $\langle\tilde{W}| = \langle W|D$ and $DE = D+E$ in the last equality. Combining equations (A.12) and (A.13), and using equation (16), we conclude that

$$\begin{aligned}H_L \begin{pmatrix} \langle\tilde{W}|E\mathcal{S}_{L-1}|V\rangle \\ \langle\tilde{W}|D\mathcal{S}_{L-1}|V\rangle \end{pmatrix} &= (\nu_L M_L - M_L(M_L + \omega_L)) \begin{pmatrix} \langle\tilde{W}|E\mathcal{S}_{L-1}|V\rangle \\ \langle\tilde{W}|D\mathcal{S}_{L-1}|V\rangle \end{pmatrix} \\ &= M_L \begin{pmatrix} \langle W|E\mathcal{S}_{L-1}|V\rangle \\ \langle W|D\mathcal{S}_{L-1}|V\rangle \end{pmatrix} = 0.\end{aligned} \tag{A.14}$$

The last equality is true because the vector on which M_L acts is precisely the TASEP-steady-state vector, given by the algebra (A.2)–(A.4).

A.4. Proof that the Kernel of H_L is contained in the Kernel of G_L

Using the recursive formula (17), we can write G_L as follows:

$$G_L = \kappa_L M_L + \begin{pmatrix} 0 & 0 \\ 0 & G_{L-1} \end{pmatrix} \qquad \text{with} \qquad \kappa_L = \begin{pmatrix} \mathbb{1}_{L-1} & \mathbb{1}_{L-1} \\ 0 & 0 \end{pmatrix}. \tag{A.15}$$

From equation (A.11), we deduce that $(\kappa_L M_L)h_L = 0$, and this would remain true if $\langle\tilde{W}|$ were replaced by any bra-vector. The action of G_L on h_L is thus given by

$$G_L \begin{pmatrix} \langle\tilde{W}|E\mathcal{S}_{L-1}|V\rangle \\ \langle\tilde{W}|D\mathcal{S}_{L-1}|V\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & G_{L-1} \end{pmatrix} \begin{pmatrix} \langle\tilde{W}|E\mathcal{S}_{L-1}|V\rangle \\ \langle\tilde{W}|D\mathcal{S}_{L-1}|V\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ G_{L-1}\langle\tilde{W}|D\mathcal{S}_{L-1}|V\rangle \end{pmatrix}. \tag{A.16}$$

This identity is satisfied regardless of the precise form of the bra-vector $\langle \tilde{W}|$. Finally, rewriting the string $\mathcal{S}_{L-1} = \binom{E\mathcal{S}_{L-2}}{D\mathcal{S}_{L-2}}$ and defining $\langle \tilde{\tilde{W}}| = \langle \tilde{W}|D$, the last equation can be recast in a form that makes its recursive structure clear:

$$G_L h_L = \begin{pmatrix} 0 \\ G_{L-1} \begin{pmatrix} \langle \tilde{\tilde{W}}|E\mathcal{S}_{L-2}|V\rangle \\ \langle \tilde{\tilde{W}}|D\mathcal{S}_{L-2}|V\rangle \end{pmatrix} \end{pmatrix}. \tag{A.17}$$

Using iteratively equation (A.16), we conclude that $G_L h_L = 0$, i.e. that the kernel of H_L is included in that of G_L and that, therefore, equation (15) is solvable.

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